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Asymptotics for a $\mathbb{Q}$-tensor model of liquid crystals

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Rims Proceddings: Mathematical Analysis of viscous incompressible fluids

Abstract

In this note we give a resumé of results that have been published in [1]. Consideration is given to a $\mathbb{Q}$-tensor model for the motion of a viscous incompressible liquid crystal flow in $\mathbb{R}^3$. The asymptotic behavior of weak solutions is studied. Sketches of the proof are presented, details can be found in [1]

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1 Introduction

The main phases in which liquid crystals are usually studied are the Nematic, Smectic and Chiral. ones. Briefly these phases can be described as folllows:

1. Nematic Phase: The main characteristic of the nematic phase is the absence of a positional order of the molecules. On the other hand the molecules do align following some range directions, positioning themselves so that there principal axes are relatively parallel. Aligned nematics have the optical properties of uniaxial crystals and are very useful in liquid crystal displays,

2. Smectic Phase They are found at lower temperatures than the nematic, present layers that can slide over one another.

3. Chiral Phase This phase has no internal planes of symmetry. It is frequently referred to the cholesteric phase.

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Liquid crystals were seen first in cholesterol derivatives. It was noticed by F. Reinitzer that these "liquids" had two melting points, and O. Lehman noticed that one could see a little crystals, and gave them the name of Liquid crystals.

This resumé deals only with nematic liquid crystal systems. Nematic models can be uniaxial and biaxial. The Oseen type models have this uniaxial main direction represented by a so called director vector, while the de Gennes model and the Maier-Saupe models give the local configuration of the crystals through a symmetric traceless $\mathbb{Q}$-tensor, admiting the possibility of biaxial representations. In this report we work with liquid crystals which are structured by a such a $\mathbb{Q}$-tensor. For specific information on the model we use we refer the reader to the papers by Zarnescu et al. [5], [6], [4]. What we describe in this paper is mostly a resumé of the paper [1], and as such for details of proofs we refer the reader to [1].

The main models for nematic liquid crystals where the representation of the orientation of the the crystals is given by a direction vector were introduced in the papers by Ericksen and Leslie [2], [3] and as pointed out in their papers " Liquid crystals are states of matter which are capable of flow, and in which the molecular arrangements give rise to a preferred direction"

The system of interest for this resumé is given by the following $\mathbb{Q}$-tensor equations, coupled with an incompressible Navier-Stokes system influenced by the behavior of the $\mathbb{Q}$-tensor :

$$\begin{aligned} \partial_t \mathbb{Q} + \mathrm{div}_x(\mathbb{Q}\mathbf{u}) - \mathcal{S}(\nabla_x \mathbf{u}, \mathbb{Q}) &= \Delta_x \mathbb{Q} - \mathcal{L}[\partial F(\mathbb{Q})], \\ \partial_t \mathbf{u} + \mathrm{div}_x(\mathbf{u} \otimes \mathbf{u}) + \nabla_x p &= \Delta_x \mathbf{u} + \mathrm{div}_x \Sigma(\mathbb{Q}) \\ \mathrm{div}_x \mathbf{u} &= 0. \end{aligned} \tag{1.1}$$

where

$$\mathcal{L}[\mathbb{A}] \equiv \mathbb{A} - \frac{1}{3}\mathrm{tr}[\mathbb{A}]\mathbb{I}$$

denotes the projection onto the space of traceless matrices, and F denotes a potential function which will be described later. In the above equations the tensors $\mathbb{S}$ and $\Sigma(\mathbb{Q})$ are given by

$$\mathbb{S}(\nabla_x \mathbf{u}, \mathbb{Q}) = (\xi\varepsilon(\mathbf{u}) + \omega(\mathbf{u}))\left(\mathbb{Q} + \frac{1}{3}\mathbb{I}\right) + \left(\mathbb{Q} + \frac{1}{3}\mathbb{I}\right)(\xi\varepsilon(\mathbf{u}) - \omega(\mathbf{u})) - 2\xi\left(\mathbb{Q} + \frac{1}{3}\mathbb{I}\right)\mathbb{Q} : \nabla_x \mathbf{u},$$

$$\Sigma(\mathbb{Q}) = 2\xi\mathbb{H} : \mathbb{Q}\left(\mathbb{Q} + \frac{1}{3}\mathbb{I}\right) - \xi\left[\mathbb{H}\left(\mathbb{Q} + \frac{1}{3}\mathbb{I}\right) - \left(\mathbb{Q} + \frac{1}{3}\mathbb{I}\right)\mathbb{H}\right] - (\mathbb{Q}\mathbb{H} - \mathbb{H}\mathbb{Q}) - \nabla_x \mathbb{Q} \odot \nabla_x \mathbb{Q},$$

where

$$\omega(\mathbf{u}) = \frac{1}{2}(\nabla_x \mathbf{u} - \nabla_x^t \mathbf{u}), \quad \mathbb{H} = \Delta_x \mathbb{Q} - \mathcal{L}[\partial F(\mathbb{Q}), \quad (\nabla_x \mathbb{Q} \odot \nabla_x \mathbb{Q})_{ij} = \partial_i \mathbb{Q}_{\alpha\beta} \partial_j \mathbb{Q}_{\alpha\beta}. \tag{1.2}$$

and the scalar parameter $\xi \geq 0$ measures the ratio between the rotation and the aligning effect that a shear flow exerts over the directors

Notation:

1 We denote by L^p, W^m_q the standard Lebesque, Sobolev spaces on $\mathbb{R}^3$, respectively.

2. $(\mathcal{R}) = (\mathbb{R}^3, \mathbb{R}^{3\times 3})$,

3. $(\mathcal{R}_0) = (\mathbb{R}^3, \mathbb{R}^{3\times 3}_{\mathrm{sym},0})$

4. $(\mathcal{R}_3) = (\mathbb{R}^3, \mathbb{R}^{3\times 3\times 3})$

Assumptions on the potential function F (APF)

1 Let $\mathcal{O} \subset \mathbb{R}^{3\times 3}_{\mathrm{sym}}$ be an open set that contains the isotropic state $\mathbb{Q} \equiv 0$,

2. Let B_{r_1}, B_{r_2} be two balls satisfying

$$\mathbb{Q} = 0 \in B_{r_1} \equiv \{|\mathbb{Q}| < r_1\} \subset B_{r_2} \equiv \{|\mathbb{Q}| \leq r_2\} \subset \mathcal{O}$$

3. $F \in C^2(\mathcal{O})$,

4. F has a unique global minimum $\mathbb{Q} = 0$ in $\mathcal{O}$, satisfying

$$F(0) = 0, \;\; F(\mathbb{Q}) > 0 \text{ for any } \mathbb{Q} \in \mathcal{O} \setminus \{0\}, \quad \text{and} \tag{1.3}$$

$$\partial F(\mathbb{Q}) \quad \mathbb{Q} \geq 0 \text{ whenever } \mathbb{Q} \in B_{r_1} \text{ or } \mathbb{Q} \in \mathcal{O} \setminus B_{r_2} \tag{1.4}$$

Remark 1.1. *The main potential of interest here are the polynomial considered by Paicu and Zarnescu [5]*

$$F(\mathbb{Q}) = \frac{a}{2}|\mathbb{Q}|^2 + \frac{b}{3}\,\mathrm{trace}[\mathbb{Q}^3] + \frac{c}{4}|\mathbb{Q}|^4, \tag{1 5}$$

Remark 1.2. *Note that, at least in the case when $a > 0$ in a neighborhood of 0, the potential* (1.5) *satisfies our conditions For more details the reader is referred to [6], [5]*

2 Energy balance

Let the energy be given by

$$E = \frac{1}{2}|\mathbf{u}|^2 + \frac{1}{2}|\nabla_x \mathbb{Q}|^2 + F(\mathbb{Q}),$$

In the sequel we suppose that if $\mathbf{u} \to 0, \mathbb{Q} \to 0$ for $|x| \to \infty$. We recall the following energy balance equation obtained as follows:
Multiply the $\mathbb{Q}$-tensor equation by $\Delta_x \mathbb{Q} - \mathcal{L}[\partial F(\mathbb{Q})]$ and the NS equation by $\mathbf{u}$ then after integrating in space we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^3} E \ \mathrm{d}x + \int_{\mathbb{R}^3} |\nabla_x \mathbf{u}|^2 + \left| \Delta_x \mathbb{Q} - \mathcal{L}[\partial F(\mathbb{Q})] \right|^2 \ \mathrm{d}x = 0.$$

The dissipative term in the above equations, suggests the following asymptotic behavior:

$$\nabla_x \mathbf{u}(t, \cdot) \to 0, \ \mathbb{Q}(t, \cdot) \to \tilde{\mathbb{Q}} \text{ as } t \to \infty,$$

where $\tilde{\mathbb{Q}}$ is a static distribution of the $Q-$tensor density, satisfying

$$-\Delta_x \tilde{\mathbb{Q}} + \mathcal{L}\left[\partial F(\tilde{\mathbb{Q}})\right] = 0.$$

The plan now is to show for weak solutions that

1. $\tilde{\mathbb{Q}} = 0$
2. $\nabla_x \mathbf{u}(t, \cdot) \to 0, \ \mathbb{Q}(t, \cdot) \to 0$ as $t \to \infty$ in a certain sense,
3. $\|\mathbf{u}(t, \cdot)\|_{L^2(\mathbb{R}^3;\mathbb{R}^3)} + \|\mathbb{Q}(t, \cdot)\|_{H^1(\mathbb{R}^3;\mathbb{R}^{3\times 3})} \leq c(1+t)^{-3/4}$

In the sequel we suppose the following simplifying assumption: $\xi = 0$. Hence the tensors $\mathbb{S}(\nabla_x \mathbf{u}, \mathbb{Q})$ and $\Sigma(\mathbb{Q})$ are reduced to

$$\begin{aligned} \mathbb{S}(\nabla_x \mathbf{u}, \mathbb{Q}) &= \omega(\mathbf{u})\mathbb{Q} - \mathbb{Q}\omega(\mathbf{u}), \\ \Sigma(\mathbb{Q}) &= -\mathbb{Q}\Delta_x \mathbb{Q} + \Delta_x \mathbb{Q}\mathbb{Q} - \nabla_x \mathbb{Q} \odot \nabla_x \mathbb{Q}, \end{aligned} \tag{2.1}$$

where $\omega(\mathbf{u})$ and $(\nabla_x \mathbb{Q} \odot \nabla_x \mathbb{Q})_{ij}$ were specified in (1.2).

3 Existence of weak solutions

We recall that a rigorous proof of *existence* for weak solutions to the (1.1) equations in the space $\mathbb{R}^3$ was established by Paicu and Zarnescu [5], with potentials satisfying conditions as specified above in (APF). The decay work in this resumé for such a class of weak solutions. Specifically we suppose that for
Data conditions (DC): $\mathbf{u}_0 \in L^1 \cap L^2(\mathbb{R}^3;\mathbb{R}^3)$, $\mathrm{div}_x \mathbf{u}_0 = 0$, $\mathbb{Q}_0 \in L^1 \cap W^{1,2}(\mathcal{R}_0)$, $|\mathbb{Q}_0(x)| \leq r_2$ for a.a. $x \in \mathbb{R}^3$
Our solution $(\mathbb{Q}, \mathbf{u})$ satisfies:

(a) $\mathbb{Q} \in C_{\text{weak}}([0,T]; L^2(\mathcal{R}))$,
$\sup_{t\in[0,T]} \left(\|\mathbb{Q}(t,\cdot)\|_{L^1\cap L^\infty(\mathcal{R})} + \|\mathbb{Q}\|_{W^{1,2}(\mathcal{R})}\right) < \infty$,
$\mathbb{Q} \in L^2(0,T; W^{2,2}(\mathcal{R}))$,
$\mathbb{Q}(t,x) \in B_{r_2}\ \forall\, t \in [0,T]$, a.a.$x \in \mathbb{R}^3$, for any $T > 0$

(b) $\mathbf{u} \in C_{\text{weak}}(0,T; L^2(\mathbb{R}^3;\mathbb{R}^3))$, $\nabla_x \mathbf{u} \in L^2(0,T; L^2(\mathcal{R}))$ for any $T > 0$.

Moreover the Paicu-Zarnescu solutions satisfy the energy inequality

$$E(t) + \int_s^t \int_{\mathbb{R}^3} \left[|\nabla_x \mathbf{u}|^2 + \left| \Delta_x \mathbb{Q} - \mathcal{L}[\partial F(\mathbb{Q})] \right|^2 \right] \,\mathrm{d}x \leq E(s) \tag{3.1}$$

$\forall\, t > s$ for $s = 0$ and a.a. $s \in [0,\infty)$.

4 Main result

Theorem 4.1. *Let F satisfy the assumptions (APF) and the initial data conditions (DC) specified before. Let $(\mathbb{Q}, \mathbf{u})$ be a global-in-time weak solution of the system* (1.1) *with such data and potential, which satisfies the energy inequality* (3.1) *then there exists $C = C(\mathbf{u}_0, \mathbb{Q}_0) > 0$ so that*

$$\|\mathbf{u}(t,\cdot)\|_{L^2(\mathbb{R}^3;\mathbb{R}^3)} + \|\mathbb{Q}(t,\cdot)\|_{W^{1,2}(\mathbb{R}^3;\mathbb{R}^{3\times 3})} \leq C(1+t)^{-\frac{3}{4}} \tag{4.1}$$

Proof *Sketch of the proof is presented, for details see [1]*

Step 1: Decay of the $\mathbb{Q}-$ tensor and L^∞ bound
To establish the decay of the $\mathbb{Q}-$*tensor* and an L^∞ bound multiply $\mathbb{Q}$-tensor equation by $2G'(|\mathbb{Q}|^2)\mathbb{Q}$, $G' \in C[0,\infty)$, and integrate

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3} G(|\mathbb{Q}|^2)\ \mathrm{d}x + \int_{\mathbb{R}^3}\left[2G'(|\mathbb{Q}|^2)|\nabla_x\mathbb{Q}|^2 + G''(|\mathbb{Q}|^2)\left|\nabla_x|\mathbb{Q}|^2\right|^2\right]\ \mathrm{d}x \\ = -2\int_{\mathbb{R}^3} G'(|\mathbb{Q}|^2)\partial F(\mathbb{Q}) : \mathbb{Q}\ \mathrm{d}x. \tag{4.2}$$

To get the L^∞ bound for $\mathbb{Q}$ use the above equation with an appropriate choice of G. Specifically it can be shown that if the initial data is in the ball B_{r_2}, then the solution stays in B_{r_2} for all time. By hypothesis $|\mathbb{Q}_0| < r_2$. Choose

$$G(z) = 0, z \in [0, r_2^2],\ G' \geq 0, G'' \geq 0,\ G(z) > 0, z > r_2^2.$$

Then

$$\int_{\mathbb{R}^3} G(|\mathbb{Q}|^2)(t,\cdot)\ \mathrm{d}x \leq \int_{\mathbb{R}^3} G(|\mathbb{Q}_0|^2)\ \mathrm{d}x = 0 \text{ for all } t \geq 0,$$

From the above, and the hypothesis on F it follows that

$$0 \leq F(\mathbb{Q}) \leq \alpha|\mathbb{Q}|^2,\ \alpha > 0. \tag{4.3}$$

Hence from the class of regularity of the solution and the energy inequality (3.1) we have

$$\sup_{t\geq 0}\left[\|\mathbb{Q}(t)\|_{L^\infty(\mathcal{R})} + \|F(\mathbb{Q}(t))\|_{L^1(\mathbb{R}^3)} + \|\nabla_x\mathbb{Q}(t)\|_{L^2(\mathcal{R}_3)}\right] \leq c.$$

Step 3: L^∞ decay to zero of $\mathbb{Q}$
The energy estimate (3.1) insures that as $n \to \infty$

$$\Delta_x\mathbb{Q}(t_n,\cdot) - \mathcal{L}[\partial F(\mathbb{Q})](t_n,\cdot) = g_n \to 0 \text{ in } L^2(\mathbb{R}^3;\mathbb{R}^{3\times 3}).$$

Now combine the last expression with appropriate bounds to show that $\mathbb{Q}(t_n,\cdot) \to 0$ uniformly. The following lemma is needed:

Pochožaev's style Lemma Let $F \in C^2(B_{r_2})$, $F(0) = 0$. Let $\mathbb{Q}$ be a solution to

$$-\Delta\mathbb{Q} + \mathcal{L}[\partial F(\mathbb{Q})] = 0 \text{ in } \mathbb{R}^3$$

wich satisfies $|\mathbb{Q}| \leq r_2$, $|\nabla_x\mathbb{Q}|^2$, $F(\mathbb{Q}) \in L^1(\mathbb{R}^3)$. Then $\mathbb{Q}$ satisfies Pochožaev's identity

$$\int_{\mathbb{R}^3}\left(\frac{1}{2}|\nabla_x\mathbb{Q}|^2 + 3F(\mathbb{Q})\right)\mathrm{d}x = 0.$$

In particular, $\mathbb{Q} \equiv 0$ provided that $F \geq 0$ in B_{r_2}.

To proof the Pochožaev's style Lemma, multiply the $\mathbb{Q}$ equation in (1.1) by $\mathbf{x}\cdot\nabla\mathbb{Q}$, integrate and use the divergence Theorem

Shift $\mathbb{Q}(t_n,\cdot)$ in x so that $|\mathbb{Q}(t_n,0)| \geq \frac{1}{2}\sup_{x\in\mathbb{R}^3}|\mathbb{Q}(t_n,x)|$. Since at least for a suitable subsequence satisfies

$$\mathbb{Q}(t_n,0) \to \tilde{\mathbb{Q}} \text{ in } C_{\mathrm{loc}}(\mathbb{R}^3;\mathbb{R}^{3\times 3}),$$

By Pochožaev's style Lemma , $\tilde{\mathbb{Q}} = 0$. Hence at least for a suitable subsequence

$$\|\mathbb{Q}(t_n,\cdot)\|_{L^\infty(\mathbb{R}^3;\mathbb{R}^{3\times 3})} \to 0 \text{ as } t_n \to \infty, \tag{4.4}$$

1. Using the same arguments that showed that $|\mathbb{Q}(t,\cdot)| < r_2$ we can show $|\mathbb{Q}(t,\cdot)| < r_1$ for all $t \gg 1$.

2. Let $G(z) = \left[\left(z - \frac{r_1^2}{4}\right)_+\right]^2$, By sequential decay there exists $\tilde{t}$ so that $\|\mathbb{Q}(\tilde{t},\cdot)\|_{L^\infty} \leq r_1/2$ then repeating the argument with this new G yields $\forall t \geq \tilde{t} \Rightarrow \|\mathbb{Q}(t,\cdot)\|_{L^\infty} \leq r_1/2$
3. The same argument now gives

$$\|\mathbb{Q}(t,\cdot)\|_{L^\infty(\mathbb{R}^3,\mathbb{R}^{3\times 3})} \to 0 \text{ as } t \to \infty.$$

Step 3: L^1 bound for $\mathbb{Q}$ and, L^2 decay of $\mathbb{Q}$

1. Choose $G(z) = \sqrt{z}$, since $G'(|\mathbb{Q}|^2)|\nabla_x\mathbb{Q}|^2 + G''(|\mathbb{Q}|^2)\left|\nabla_x|\mathbb{Q}|^2\right|^2 \geq 0$, by (4.2)

$$\|\mathbb{Q}(t,\cdot)\|_{L^1(\mathbb{R}^3;\mathbb{R}^{3\times 3})} \leq C \text{ for all } t > 0.$$

2. L^2 decay for $\mathbb{Q}$ Choose $G(z) = z$ then

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}|\mathbb{Q}|^2\,\mathrm{d}x + \int_{\mathbb{R}^3}|\nabla_x\mathbb{Q}|^2\,\mathrm{d}x \leq 0 \text{ for all } t \gg K.$$
$$\|\mathbb{Q}\|_{L^2(\mathcal{R})} \leq \|\mathbb{Q}\|_{L^1(\mathcal{R}))}^{2/5}\|\mathbb{Q}\|_{L^6(\mathcal{R})}^{3/5} \leq c\|\nabla_x\mathbb{Q}\|_{L(\mathcal{R}_3)}^{3/5}.$$

Combining the last two inequalities yields $\|\mathbb{Q}(t,\cdot)\|_{L^2(\mathcal{R})}^2 \leq c(1+t)^{-3/2}$ for all $t \geq 0$.

Step 4: Energy computations and L^2 decay for u

Remark 4.1. *By* (4.3) *it follows that* $\int_{\mathbb{R}^3}|\mathcal{L}[\partial F(\mathbb{Q})]|^2\,\mathrm{d}x \leq c\int_{\mathbb{R}^3}|\mathbb{Q}|^2\,\mathrm{d}x$, Let $\mathcal{E}(t) = \int_{\mathbb{R}^3} \frac{1}{2}|\mathbf{u}|^2 + \frac{1}{2}|\nabla_x \mathbb{Q}|^2 + F(\mathbb{Q})dx$. An easy computation shows

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t) + \int_{\mathbb{R}^3} \left[|\nabla_x \mathbf{u}|^2 + \frac{1}{2}|\Delta_x \mathbb{Q}|^2 - c\,|\mathcal{L}[\partial F(\mathbb{Q})]|^2 \right] \,\mathrm{d}x \leq 0.$$

Hence by the L^2 decay of the $\mathbb{Q}$ tensor it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t) + \int_{\mathbb{R}^3} \left[|\nabla_x \mathbf{u}|^2 + \frac{1}{2}|\Delta_x \mathbb{Q}|^2 \right] \,\mathrm{d}x \leq c(1+t)^{-3/2}, \tag{4.5}$$

The L^2 decay for $\mathbf{u}$ follows by a modified version of the Fourier Splitting method [7]. Since one can show that

$$\int_{\mathbb{R}^3} |\Delta \mathbb{Q} - \mathcal{L}\left[\partial F(\mathbb{Q})\right]|^2 \,\mathrm{d}x \geq c \int_{\mathbb{R}^3} \left[\frac{1}{2}|\nabla_x \mathbb{Q}|^2 + F(\mathbb{Q}) \right] \,\mathrm{d}x, \; c > 0,$$

by (4.5) and the last inequality give

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t) + \int_{\mathbb{R}^3} \left[|\xi \widehat{u}|^2 + \frac{1}{2}|\nabla_x \mathbb{Q}|^2 + F(\mathbb{Q}) \right] \,\mathrm{d}x \leq c(1+t)^{-3/2},$$

It is on this last energy inequality where the Fourier splitting method is applied to obtain first an auxiliary rate of decay. Use the splitting $\int_{\mathbb{R}^3} |\xi \widehat{\mathbf{u}}|^2 \,\mathrm{d}\xi \geq R^2(t) \int_{|\xi| \geq R(t)} |\widehat{\mathbf{u}}|^2 \,\mathrm{d}\xi$. Then for any $R(t) \in [0,1]$ the last inequality yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t) + R^2(t)\mathcal{E}(t) \leq R^2(t) \int_{|\xi| < R(t)} \left|\widehat{\mathbf{u}}\right|^2 \,\mathrm{d}\xi + c(1+t)^{-3/2}$$

The Fourier Transform of velocity is given by

$$\widehat{u}_i(t,\xi) = \exp\left(-|\xi|^2 t\right) \widehat{u}_{0,i}(\xi) + \int_0^t \exp\left(-|\xi|^2(t-s)\right) \sum_{j,k} \left[\left(\delta_{i,j} - \frac{\xi_i \xi_j}{|\xi|^2} \right) \xi_k \tilde{\mathbb{Q}}(s,\xi) \right] \,\mathrm{d}s.$$

where $\tilde{\mathbb{Q}}(s,\xi) = \left(-\widehat{\mathbb{Q}\Delta_x \mathbb{Q}} + \widehat{\Delta_x \mathbb{Q}\mathbb{Q}} - \nabla_x \widehat{\mathbb{Q} \odot \nabla_x} \mathbb{Q} - \widehat{\mathbf{u} \otimes \mathbf{u}} \right)_{j,k} (s,\xi)$.

Step 5: First auxiliary estimates
An easy computation shows $\tilde{\mathbb{Q}}(s,\xi) \leq |\xi|(1+|\xi|)\left(\mathcal{E}(s) + (1+s)^{-3/2}\right)$

Let $\mathcal{M}(s) = r^2\left(\mathcal{E}(s) + (1+s)^{-3/2}\right) \mathrm{d}s$ then $\forall\, 0 \le R(t) \le 1$, combining the last inequalities gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t) + R^2(t)\mathcal{E}(t) = c(t+1)^{-3/2} + c\left[R^2(t)\int_0^{R(t)}\left(\int_0^t \exp\left(-r^2(t-s)\right)\mathcal{M}(s)\right)^2 \mathrm{d}r\right] \tag{4.6}$$

Inequality (4.6) is the starting point for a bootstrap procedure to deduce the desired decay estimate (4.1). The following auxiliary Lemma will be needed

Lemma 4.1. *Let $\gamma \in (0,1)$, $\mu > 0$ and $\gamma < \mu$. If $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t) + (1+t)^{-\gamma}\mathcal{E}(t) \le c(1+t)^{-\mu}$, then*

$$\mathcal{E}(t) \le c(\gamma, \mathcal{E}(0))(1+t)^{-\mu+\gamma}.$$

Now combine the last lemma and (4.6) and start the bootstrap argument . Suppose it has been already shown

$$\mathcal{E}(t) \le c(1+t)^{-\alpha},\ 0 \le \alpha. \tag{4.7}$$

Then one can show that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t) + R^2(t)\mathcal{E}(t) \le \begin{cases} c(\alpha)\left[(t+1)^{-3/2} + R^3(t)\right], & \alpha = 0, \\ c(\alpha)\left[(t+1)^{-3/2} + R^7(t)\right], & \alpha > 0,\ \alpha \ne 1. \end{cases}$$

Since at least for $\alpha = 0$ inequality (4.7) holds we can begin the bootstrap argument. Let $R(t) = (1+t)^{-\beta}$ and

- Use auxiliary Lemma with $\beta = \frac{1}{2} - \frac{\epsilon}{3}$ to get $\mathcal{E}(t) \le C(t+1)^{-\frac{15}{14}}$.
- This first decay then is used to get a better estimate of $\int_{|\xi|<R(t)} |\widehat{u}|^2 d\xi$.
- Obtain estimate that with $R(t) = 1$ yields
$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t) + \mathcal{E}(t) \le Ct^{-3/2}.$$
- Last inequality yields $\mathcal{E}(t) \le C(t+1)^{-\frac{3}{2}}$.

This completes the sketch of the proof of the main theorem. $\square$

5 Example

Polynomial potentials (Paicu and Zarnescu):

$$F(\mathbb{Q}) = \frac{a}{2}|\mathbb{Q}|^2 + \frac{b}{3}\,\mathrm{trace}[\mathbb{Q}^3] + \frac{c}{4}|\mathbb{Q}|^4, \tag{5.1}$$

- At least in case $a > 0$ in a neighborhood of 0, fit our conditions
- there exists an open neighborhood $\mathcal{O}$ of 0 such that F satisfies the hypotheses of decay Theorem.
- Hence for weak solution with potentials of the type given by (5.1) it follows that

$$\|\mathbf{u}(t,\cdot)\|_{L^2(\mathcal{R})} + \|\mathbb{Q}(t,\cdot)\|_{W^{1,2}(\mathcal{R})} \leq c(1+t)^{-3/4}$$

Finally by an induction argument the following optimal L^p decay of the $\mathbb{Q}$ can be obtained

Theorem 5.1. *Let F and the initial data satisfy the assumptions from Theorem 4.1 and suppose additionally that $\mathbb{Q}_0 \in L^q, 1 \leq q \leq \infty$. Let $(\mathbb{Q}, \mathbf{u})$ be a global-in-time weak solution of the system* (1.1) *satisfying the energy inequality* (3.1) *with the given data and potential F. Then for all $t > 0$, and all $q \geq 1$ it follows that*

$$\|\mathbb{Q}\|_q \leq C_q(1+t)^{-\frac{3}{2q'}}, \tag{5.2}$$

where $\frac{1}{q} + \frac{1}{q'} = 1$. The constant C depends on the L^q and the $L^{\frac{q}{2}}$ norms of the data

Proof We only give the main ideas. The details will appear in a forthcoming paper [8]. Inequality (5.2) has been established for $q = 2$. Suppose it holds for $q = k$ for some even $k > 2$. Let $G(z) = z^k$, in (4.2). Since the right hand side of (4.2) is negative and $2G'(|\mathbb{Q}|^2)|\nabla_x\mathbb{Q}|^2 + G''(|\mathbb{Q}|^2)\left|\nabla_x|\mathbb{Q}|^2\right|^2$ is positive it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\frac{1}{2}|\mathbb{Q}|^{2k}\,\mathrm{d}x + \int_{\mathbb{R}^3}|\nabla_x\mathbb{Q}|^{2(k-1)}|\nabla\mathbb{Q}|^2\,\mathrm{d}x \leq 0 \text{ for all } t > 0 \text{ large enough.} \tag{5.3}$$

The proof now for is obtained by steps similar to the ones used for the L^2 decay. The decay for solutions in L^q-spaces with non even powers is obtained by simple interpolation. Finally for $q = \infty$ it follows passing to the limit. $\square$

References

[1] M. Dai, E. Feireisl, E. Rocca, G. Shimperna and M. Schonbek. On asymptotic isotropy for a hydrodynamic model of liquid crystals. *arXiv:1409.7499. Accepted for publication in Asymptotic Analysis.*

[2] F. M. Leslie. *Some Contitutive Equations for liquid crystals.* Arch Rational Mech Anal. 28 (1968) 265 - 283.

[3] F. M. Leslie. *Theory of flow phenomena in liquid crystals.* Advances in Liquid Crystals, Vol 4 G. Brown ed., Academic Press, New York, 1979 1- 81.

[4] E. Kirr, M. Wilkinson, and A. Zarnescu. Dynamic Statistical Scaling in the Landau–de Gennes Theory of Nematic Liquid Crystals. *J. Stat. Phys.*, **f** 155:625–657, 2014.

[5] M. Paicu and A. Zarnescu. Global existence and regularity for the full coupled Navier-Stokes and Q-tensor system. *SIAM J. Math. Anal.*, **f** 43(5):2009–2049, 2011.

[6] M. Paicu and A. Zarnescu. Energy dissipation and regularity for a coupled Navier-Stokes and Q-tensor system. *Arch. Ration. Mech. Anal.*, **f** 203(1):45–67, 2012.

[7] M. Schonbek. L^2 decay for weak solutions of the Navier-Stokes equations. *Arch. Ration. Mech. Anal.*, f88 (3), pp. 209-22, 1985.

[8] M. Schonbek and Y Shibata On a global well-posedness for a model of a $\mathbb{Q}$ tensor model of Incompressible Nematic Liquid Crystals in $\mathbb{R}^3$ Forthcoming